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The Entropy of Partitions on MV-Algebras†

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Partitions of MV-algebras are studied. Using the notion of a state (as a probabilistic measure) on MV-algebras, we introduce the entropy of partitions. We show a suitable method for the refinement of partitions and the subadditivity of the entropy with respect to this refinement.

1. INTRODUCTION

The entropy of partitions on probabilistic spaces was introduced by Kolmogorov and Sinaj [6, 10] as a useful tool for studying the isomorphism of dynamical systems. In recent years the entropy of partitions has been applied in many other structures. For example, for an overview of several types of entropy in the fuzzy environment see ref. 5. Common sketches of the entropy on some structures were generalized by Riečan and Neubrunn [9] using the notion of an algebraic entropy. In this paper we will investigate the entropy of partitions on MV-algebras.

2. PRELIMINARIES

MV-algebras were introduced by Chang [1] (see also ref. 4). There are often used as an algebraic model for many-valued logics.

Definition 1 [1, 4]. An algebra $\{\mathcal{M}, \mathbf{0}, \mathbf{1}, ', \oplus \odot\}$ is said to be an *MV-algebra* iff it satisfies the following conditions:

(MV1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ (*associativity*)
(MV2) $x \oplus y = y \oplus x$ (*symmetry*)
(MV3) $x \oplus \mathbf{0} = x$ (*neutral element*)

† This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.
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(MV4) $x \oplus \mathbf{1} = \mathbf{1}$ (*annihilator*)
(MV5) $\mathbf{0}' = \mathbf{1}$ and $\mathbf{1}' = \mathbf{0}$ (*boundary conditions*)
(MV6) $x \odot y = (x' \oplus y')'$ (*De Morgan law*)
(MV7) $y \oplus (y \oplus x')' = x \oplus (x \oplus y')'$ (*compatibility*)

Note that (MV7) ensures $(x')' = x$ for all $x \in \mathcal{M}$, i.e., the complementation $': \mathcal{M} \to \mathcal{M}$ is an involutive mapping. In addition, by the De Morgan law (MV6), the operation $\odot$ is associative and symmetric, with the neutral element **1** and annihilator **0**.

Further, the compatibility allows us to introduce the *lattice structure* on $\mathcal{M}$:

$$x \vee y = x \oplus (x \oplus y')', \qquad x \wedge y = (x \oplus y') \odot y$$

and the *partial order* $\leq$:

$$x \leq y \quad \text{iff} \quad x \vee y = y$$

Any MV-algebra is a distributive lattice with respect to the operations $\vee$ and $\wedge$.

For more details and other properties of MV-algebras see the overview by Cignoli *et al.* [4].

Example 1. Let $\mathcal{I}$ be a subset of the interval [0, 1] of real numbers such that $0 \in \mathcal{I}$, $1 \in \mathcal{I}$, and if $a, b \in \mathcal{I}$, then

$$a \oplus b := \min(1, a + b) \in \mathcal{I}$$

$$a \odot b := \max(0, a + b - 1) \in \mathcal{I}$$

$$a' := 1 - a \in \mathcal{I}$$

where symbols $+$ and $-$ denote the usual sum and difference of real numbers.

The system $\mathcal{I}$ is an MV-algebra. Moreover,

$$a \vee b = \max(a, b), \qquad a \wedge b = \min(a, b)$$

and the relation $\leq$ is the natural order of real numbers.

It is not difficult to show that, for $a, b \in \mathcal{I}$,

$$a \oplus b = a + b \qquad \text{if and only if} \quad a \leq b' = 1 - b$$

As an explicit example, we will take the system $\mathcal{I} = \{0, 1/n, 2/n, \ldots, 1\}$, $n \in \mathbb{N}$. Then for $a, b \in \mathcal{I}$, $a = k/n$, $b = l/n$, $k \leq n$, $l \leq n$, we obtain

$$a \oplus b = \min\left(1, \frac{k + l}{n}\right), \qquad a \odot b = \max\left(0, \frac{k + l - n}{n}\right),$$

$$a' = \frac{n - k}{n}$$

Another example is the system $\mathcal{I} = \mathbb{Q} \cap [0, 1]$, where $\mathbb{Q}$ is the set of all rational numbers.

It is known that an MV-algebra $\{\mathcal{M}, \mathbf{0}, \mathbf{1}, ', \oplus, \odot\}$ can be identified with a Boolean D-poset $\{\mathcal{M}, \mathbf{0}, \mathbf{1}, \leq, \ominus\}$ [3, 7] and the operation of *difference* ($\ominus$) is given by

$$a \ominus b = (a' \oplus b)' = a \odot b' \qquad \text{for any} \quad a, b \in \mathcal{M}$$

Definition 2. A *state* on an MV-algebra $\mathcal{M}$ is a mapping $m: \mathcal{M} \to [0, 1]$ satisfying the following conditions:

(i) $m(\mathbf{1}) = 1$.
(ii) If $a_n \nearrow a$ ($a_n \leq a_{n+1}$; $a = \vee_{n=1}^{\infty} a_n$), then $m(a) = m(a_1) + \Sigma_{n=2}^{\infty} m(a_n \ominus a_{n-1})$.

Note that a similar definition of a state on an MV-algebra was established by Chovanec [2].

Lemma 1. If $a \leq b$, then the following hold:

(i) $m(b) = m(a) + m(b \ominus a)$.
(ii) $m(b \ominus a) = m(b) - m(a)$.
(iii) $m(a) \leq m(b)$.

Proof. If $a \leq b$, then $b = a \vee b$. Using condition (ii) of Definition 2 directly, we obtain the property (i), i.e., $m(b) = m(a) + m(b \ominus a)$, and consequently the equation in (ii).

Considering that $m(b \ominus a) \geq 0$, the inequality $m(b) \geq m(a)$ follows from (i).

Lemma 2. If $a \leq b'$, then $m(a \oplus b) = m(a) + m(b)$.

Proof. Let $a \leq b'$. Take two elements a and $c = a \oplus b$. Then $c \ominus a = (a \oplus b) \ominus a = b$ [2] and $a \leq c$. Thus according to Lemma 1, we get $m(c) = m(a) + m(c \ominus a) = m(a) + m(b)$. Hence $m(a \oplus b) = m(a) + m(b)$.

Definition 3. Two elements $a, b \in \mathcal{M}$ are *orthogonal* iff $a \leq b'$, and we denote this by the symbol $a \perp b$.

A finite system $\mathbf{P} = (a_1, a_2, \ldots, a_k)$ of elements of the MV-algebra $\mathcal{M}$ will be said to be a $\oplus$-*orthogonal system* iff

$$\left(\bigoplus_{i=1}^{l} a_i\right) \perp a_{i+1} \qquad \text{for } l = 1, 2, \ldots, k - 1$$

By Lemma 2, it is obvious that for any $\oplus$-orthogonal system $\mathcal{P} \subset \mathcal{M}$ and any state m on the MV-algebra $\mathcal{M}$ it holds that

$$m\left(\bigoplus_{i=1}^{k} a_i\right) = \sum_{i=1}^{k} m(a_i)$$

3. ENTROPY OF PARTITIONS

Now we can introduce a partition on an MV-algebra. Let $\mathcal{M}$ be an MV-algebra and let m be a state on $\mathcal{M}$.

Definition 4. A system $\mathbf{P} = (a_1, a_2, \ldots, a_k) \subset \mathcal{M}$ is said to be the *partition* of $\mathcal{M}$ corresponding to the state m iff

(P1) $\mathbf{P}$ is the $\oplus$-orthogonal system.

(P2) $m\left(\bigoplus_{i=1}^{k} a_i\right) = 1.$

Note that Mundici [8] has introduced a partition on a given MV-algebra $\mathcal{M}$ idependently on a given state m. However, each Mundici partition is also a partition with respect to an arbitrary state m.

Definition 5. Let the system $\mathbf{P} = (a_1, a_2, \ldots, a_k)$ be a partition of MV-algebra $\mathcal{M}$ corresponding to a state m. Then the *entropy* of the partition $\mathbf{P}$ with respect to m is defined by

$$H_m(\mathbf{P}) = -\sum_{i=1}^{k} \varphi(m(a_i))$$

where $\varphi(x) = x \log x$, $x > 0$, with the convention $\varphi(0) = 0$.

Definition 6. We will say that a state m on MV-algebra $\mathcal{M}$ has *Bayes' Property* iff it satisfies the following condition:

Let the system $(b_1, b_2, \ldots, b_l)$ be any partition corresponding to a state m and $a \in \mathcal{M}$; then

$$m\left(\bigoplus_{j=1}^{l} (a \odot b_j)\right) = m(a)$$

Lemma 3. Let $\mathbf{Q} = (b_1, b_2, \ldots, b_l)$ be a partition or MV-algebra $\mathcal{M}$, $a \in \mathcal{M}$, and the state m has Bayes' property. Then

$$\sum_{j=1}^{l} m(a \odot b_j) = m(a)$$

Proof. Let $\mathbf{Q} = (b_1, b_2, \ldots, b_l)$ be a partition of MV-algebra $\mathcal{M}$ corresponding to a state m which has Bayes' property. First we will show that the system $(a \odot b_1, a \odot b_2, \ldots, a \odot b_l)$ is $\oplus$-orthogonal. Put $c_j = a \odot b_j$, $j = 1, 2, \ldots, l$. We need to prove that $c_1 \perp c_2$, $(c_1 \oplus c_2) \perp c_3, \ldots.$

But $c_1 \perp c_2 \Leftrightarrow c_1 \leq c_2' \Leftrightarrow a \odot b_1 \leq (a \odot b_2)'$.

According to (MV6), $(a \odot b_2)' = a' \oplus b_2'$. From the $\oplus$-orthogonality of the system $\mathbf{Q}$ we have $b_1 \leq b_2'$. Using the monotonicity of operations $\oplus$ and $\odot$, we obtain

$$a \odot b_1 \leq b_1 \leq b_2' \leq a' \oplus b_2'$$

Similarly, we will prove that $(c_1 \oplus c_2) \perp c_3$, which is equivalent to $(a \odot b_1) \oplus (a \odot b_2) \leq (a \odot b_3)'$. Seeing that $(a \odot b_3)' = a' \oplus b_3'$ and $b_1 \oplus b_2 \leq b_3'$ ($\mathbf{Q}$ is $\oplus$-orthogonal), we can write

$$(a \odot b_1) \oplus (a \odot b_2) \leq b_1 \oplus b_2 \leq b_3' \leq a' \oplus b_3'$$

The rest of the proof is obvious.

Second, for the $\oplus$-orthogonal system $\mathbf{Q}$, by Lemma 2 and Definition 6 it holds that

$$\sum_{j=1}^{l} m(a \odot b_j) = m\left(\bigoplus_{j=1}^{l} (a \odot b_j)\right) = m(a) \quad \blacksquare$$

Definition 7. Let $\mathbf{P} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{Q} = (b_1, b_2, \ldots, b_l)$ be two partitions of an MV-algebra $\mathcal{M}$ corresponding to a state m. Then the *common refinement* of these partitions will be defined as the system

$$\mathbf{P} \cup \mathbf{Q} = (a_i \odot b_j;\ a_i \in \mathbf{P},\ b_j \in \mathbf{Q},\ i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, l)$$

Lemma 5. If the state m has Bayes' property, then the system $\mathbf{P} \cup \mathbf{Q}$ is a partition of MV-algebra $\mathcal{M}$, too.

Proof. Let $\mathbf{P} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{Q} = (b_1, b_2, \ldots, b_l)$ be partitions of MV-algebra $\mathcal{M}$ corresponding to a state m. Put $c_{ij} = a_i \odot b_j$, $i = 1, 2, \ldots, k$; $j = 1, 2, \ldots, l$.

Condition (P1): The proof is similar to that in the first part of Lemma 3. Therefore the system $\mathbf{P} \cup \mathbf{Q} = (c_{ij};\ i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, l)$ is $\oplus$-orthogonal.

Condition (P2): By the Bayes' property of the state m and the $\oplus$-orthogonality of the system $\mathbf{P} \cup \mathbf{Q}$ we obtain

$$m\left(\bigoplus_{i,j=1}^{k,l} c_{i,j}\right) = m\left(\bigoplus_{i=1}^{k}\left(\bigoplus_{j=1}^{l} (a_i \odot b_j)\right)\right) = \sum_{i=1}^{k} m\left(\bigoplus_{j=1}^{l} (a_i \odot b_j)\right)$$

$$= \sum_{i=1}^{k} m(a_i) = m\left(\bigoplus_{i=1}^{k} a_i\right) = 1 \quad \blacksquare$$

Theorem 1. Let $\mathbf{P} = (a_1, a_2, \ldots, a_k)$ and $\mathbf{Q} = (b_1, b_2, \ldots, b_l)$ be partitions of MV-algebra $\mathcal{M}$ corresponding to a state m which have the Bayes' property, and $H_m(\mathbf{P})$ and $H_m(\mathbf{Q})$ be their entropies. Then

$$H_m(\mathbf{P} \cup \mathbf{Q}) \leq H_m(\mathbf{P}) + H_m(\mathbf{Q})$$

Proof. Assume that the premises of Theorem 1 are satisfied. First we will introduce the conditional state: Let $a, b \in \mathcal{M}$ and m be a state on $\mathcal{M}$. Then the *conditional state* is

$$m(a/b) = \begin{cases} 0, & m(b) = 0 \\ m(a \odot b)/m(b), & m(b) > 0 \end{cases}$$

Further, we use the convexity of the function $\varphi(x) = x \log x$, $x \geq 0$. By Jensen's inequality we have

$$\varphi\left(\sum_{j=1}^{l} \alpha_j x_j\right) \leq \sum_{j=1}^{l} \alpha_j \varphi(x_j), \qquad \text{where} \quad \sum_{j=1}^{l} \alpha_j = 1 \quad \text{and} \quad \alpha_j, x_j \in [0, 1] \tag{1}$$

Now we put $\alpha_j = m(b_j)$ and $x_j = m(a_i/b_j)$, $j = 1, 2, \ldots, l$. Then $a_j, x_j \in [0, 1]$, $\Sigma_{j=1}^{l} \alpha_j = \Sigma_{j=1}^{l} m(b_j) = 1$. Using the definition of a conditional state and Lemma 3, we can express

$$\sum_{j=1}^{l} \alpha_j x_j = \sum_{j=1}^{l} m(b_j) m(a_i/b_j) = \sum_{j=1}^{l} m(b_j) \frac{m(a_i \odot b_j)}{m(b_j)}$$

$$= \sum_{j=1}^{l} m(a_i \odot b_j) = m(a_i)$$

$$\varphi\left(\sum_{j=1}^{l} \alpha_j x_j\right) = \varphi(m(a_i)) \qquad \text{for} \quad i = 1, 2, \ldots, k$$

Similarly, we compute

$$\sum_{j=1}^{l} \alpha_j \varphi(x_j) = \sum_{j=1}^{l} m(b_j) \varphi(m(a_i/b_j)) = \sum_{j=1}^{l} m(b_j) m(a_i/b_j) \log(m(a_i/b_j))$$

$$= \sum_{j=1}^{l} m(b_j) \frac{m(a_i \odot b_j)}{m(b_j)} \log \frac{m(a_i \odot b_j)}{m(b_j)}$$

$$= \sum_{j=1}^{l} m(a_i \odot b_j)(\log m(a_i \odot b_j) - \log m(b_j))$$

$$= \sum_{j=1}^{l} m(a_i \odot b_j) \log m(a_i \odot b_j) - \sum_{j=1}^{l} m(a_i \odot b_j) \log m(b_j)$$

$$= \sum_{j=1}^{l} \varphi(m(a_i \odot b_j)) - \sum_{j=1}^{l} m(a_i \odot b_j) \log m(b_i)$$

Then the inequality (1) has the form

$$\varphi(m(a_i)) \leq \sum_{j=1}^{l} \varphi(m(a_i \odot b_j)) - \sum_{j=1}^{l} m(a_i \odot b_j) \log m(b_j)$$

for $i = 1, 2, \ldots, k$.

Summarizing these inequalities, we obtain

$$\sum_{i=1}^{k} \varphi(m(a_i)) \leq \sum_{i=1}^{k} \sum_{j=1}^{l} \varphi(m(a_i \odot b_j)) - \sum_{i=1}^{k} \sum_{j=1}^{l} m(a_i \odot b_j) \log m(b_j)$$

By Lemma 3, $\sum_{i=1}^{k} m(a_i \odot b_j) = m(b_j)$, $j = 1, 2, \ldots, l$. Therefore

$$\sum_{i=1}^{k} \sum_{j=1}^{l} m(a_i \odot b_j) \log m(b_j) = \sum_{j=1}^{l} \left(\sum_{i=1}^{k} m(a_i \odot b_j) \right) \log m(b_j) = \sum_{j=1}^{l} \varphi(m(b_j))$$

Hence

$$\sum_{i=1}^{k} \varphi(m(a_i)) \leq \sum_{i=1}^{k} \sum_{j=1}^{l} \varphi(m(a_i \odot b_j)) - \sum_{j=1}^{l} \varphi(m(b_j))$$

If we rewrite the last inequality in the language of entropies, then we have

$$-H_m(\mathbf{P}) \leq - H_m(\mathbf{P} \cup \mathbf{Q}) + H_m(\mathbf{Q})$$

$$H_m(\mathbf{P} \cup \mathbf{Q}) \leq H_m(\mathbf{P}) + H_m(\mathbf{Q}) \quad \blacksquare$$

The above results show that it is possible to introduce some notions of probability theory on MV-algebras.

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